Measure synchronization in coupled φ^4 Hamiltonian systems

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Transitions to measure synchronization both in the quasiperiodic and chaotic cases are investigated based on numerical computation of two coupled φ^4 equations. Some relevant quantities such as the bare energies, the interaction energy, and the phase difference of the two oscillators are computied to clarify the characteristics of the transitions and the measure-synchronous states. A bifurcation with discontinuous bare energy and continuous interaction energy, which takes the maximum value at the critical point, is found for the transition from the desynchronous quasiperiodic state to the measure-synchronous quasiperiodic state, and the related power law scalings are deduced. Stick-slip and random-walk-like behavior of the phase difference is found for the chaotic measure-synchronous state, and this explains the monotonous increase of the interaction energy with an increase of coupling.

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I. INTRODUCTION

Synchronization phenomena have been investigated since the 17th century [1]. The early studies focused on the synchronization of various periodic systems and, recently, chaos synchronization has attracted much attention [2–6]. Nevertheless, so far most of the works on synchronization phenomena have considered dissipative systems only, since synchronization between two trajectories is often related to the contraction of the phase space volume. Hamiltonian systems conserve their phase space volumes [7], and does not allow synchronization in the original sense, e.g., in the sense that two nonidentical trajectories approach an identical one asympototically.

In Ref. [8], the authors revealed an interesting synchronization phenomenon of Hamiltonian systems: measure synchronization (MS), i.e., they found a transition in two coupled Hamiltonian oscillators from a state where the two oscillators visit two different phase space domains (measure nonsynchronous) to a state where the two oscillators share the same phase domain (measure synchronous). Hamiltonian systems serve as typical model systems in classical as well as quantum mechanics [9–11], and a variety of practical systems can be well approximated by the Hamiltonian formalism even at weak dissipation. Thus, it is important to understand the synchronization processes of Hamiltonian systems. Moreover, it is of great significance to extend (if possible) the concept of synchronization to quantum systems (to our knowledge, no such extension has appeared), a comprehensive understanding of synchronization in Hamiltonian systems is a crucial step towards such extension.

So far we have understood much less about MS of Hamiltonian systems than synchronization behavior of dissipative systems. For instance, the behaviors of many physically important quantities have not been investigated at the MS transition. In this paper, we use two coupled Hamiltonian φ^4 equations as our model, which is a typical model in classical mechanics and can be easily formulated in quantum mechanics [11–13]. With this model we focus on the behaviors of the interaction energy and the phase angle difference between the two oscillators around the MS transition point. For the MS transition between two quasiperiodic states, we found an interesting feature that the maximum interaction energy and the maximum average phase difference are attained at the critical point, and we computed numerically the related power law scalings. This MS transition is found to be a transition with some quantities (e.g., bare energy, to be explained in text) showing a first-order discontinuous bifurcation, while some other quantities (e.g., interaction energy) showing a second-order continuous bifurcation with discontinuous slope at the transition point. For chaotic MS state, we found stick-slip phase variation and a monotonic increase of interaction energy with an increase of coupling. The paper will be organized as follows. In the following section, we present our model and describe the MS phenomenon. In Sec. III, the MS transition between quasiperiodic states is investigated and in Sec. IV the transition from quasiperiodicity to chaos through MS is studied. The conclusion and a brief discussion are given in Sec. V.

II. MODEL AND MEASURE SYNCHRONIZATION

Our model comprises two linearly coupled identical φ^4 systems with the Hamiltonian

$$H=\frac{p_1^2+p_2^2}{2}+\frac{q_1^4+q_2^4}{4}+\varepsilon(q_1-q_2)^2. \tag{1}$$

Numerically, we simulate the corresponding canonical equations

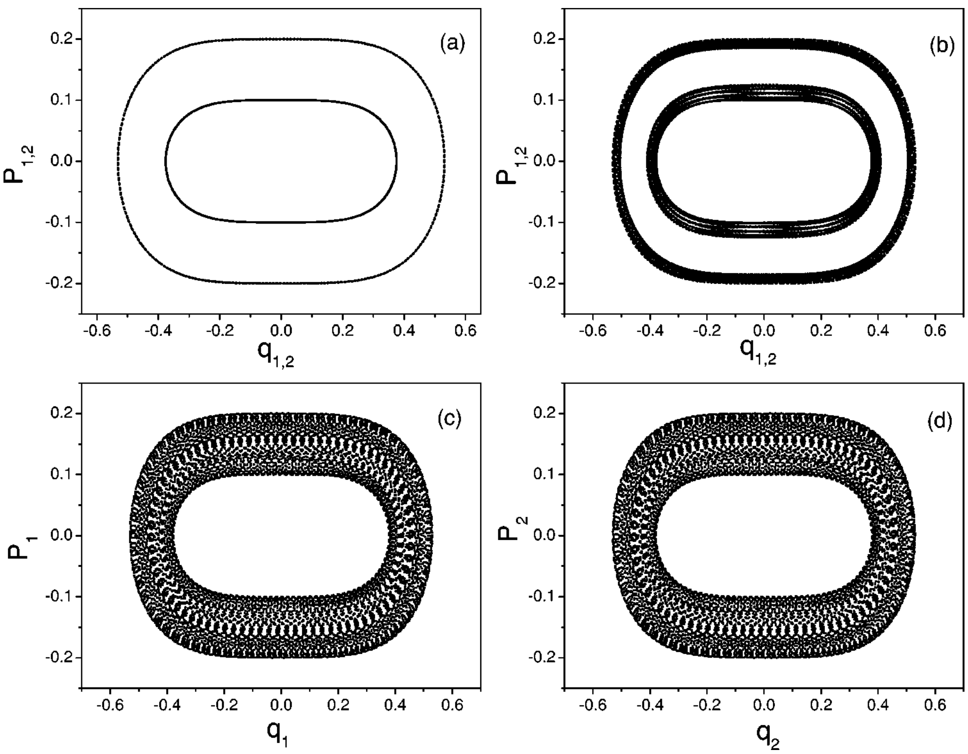

FIG. 1. Trajectories of the two oscillators of Eqs. (2) in the (q,p) plane. $p_1(0)=0.1$ and $p_2(0)=0.2$. (a) $\varepsilon=0$: The two trajectories are periodic. (b) $\varepsilon=0.001$: The two separated (measure-desynchronous) quasiperiodic orbits. (c) and (d) $\varepsilon=0.0035$: The two trajectories share the same phase space and reach MS.

$$\dot{q}_1=p_1, \quad \dot{q}_2=p_2,$$

$$\dot{p}_1=-q_1^3+2\varepsilon(q_2-q_1), \quad \dot{p}_2=-q_2^3+2\varepsilon(q_1-q_2). \tag{2}$$

A simple analysis shows that with coupling ε, serving as an adjustable parameter, the total energy can be regarded as an irrelevant parameter by a suitable scaling. Thus, we fix the total energy $H=0.025$ throughout the paper, and study the system behavior by varying the coupling ε.

The system's dynamics depends not only on ε, but also on its initial condition. Specifically, we will fix

$$q_1(t=0)=q_2(t=0)=0. \tag{3}$$

A convenient feature of this arrangement is that the interaction energy

$$E_I=\varepsilon(q_1-q_2)^2 \tag{4}$$

is kept zero initially, and the variation of ε does not change the total energy for any initial choices of $p_1(0)$ and $p_2(0)$. Therefore, in our system there are two adjustable parameters: one is the coupling strength ε and the other is the initial value of p_1 (note that $p_1^2/2\leqslant H$ and $p_2=\pm\sqrt{2H-p_1^2}$).

With system (2) we can find indeed different kinds of MS transitions. In Fig. 1 we fix $p_1(0)=0.1$, $p_2(0)=0.2$ and plot the trajectories of the two oscillators in q_j–p_j $(j=1,2)$, phase planes for different couplings. In Fig. 1(a), for zero coupling, both oscillators have periodic orbits in different energy surfaces determined by its initial conditions. With nonzero but small couplings [Fig. 1(b), $\varepsilon=0.001$], the two periodic orbits of Fig. 1(a) are replaced by two smooth quasiperiodic trajectories wandering in two distinctive tori. By further increasing ε [Figs. 1(c) and 1(d), $\varepsilon=0.0035$], we find that the two distinctive tori of Fig. 1(b) merge to an identical enlarged torus, and both the oscillators share the same phase domain with their quasiperiodic motions, indicating a phenomenon of MS. In Fig. 2, we do the same as Fig. 1 by replacing the initial condition as $p_1(0)=5.42\times10^{-2}$, $p_2(0)=0.217$, and using different couplings. The results are the same as Fig. 1, except that the trajectories in Figs. 2(c) and 2(d) are chaotic rather than quasiperiodic as in Figs. 1(c) and 1(d). Therefore, we observe a MS transition between different quasiperiodic (QP to QP), motions, and also a MS transition from quasiperiodicity to chaos (QP to CH).

In order to clarify the MS transitions we measure the average bare energies $h_j=\langle E_j\rangle$, $j=1,2$ as

$$h_{1,2}=\frac{1}{T}\int_0^T E_{1,2}(t)dt,$$

$$E_{1,2}(t)=\frac{p_{1,2}^2(t)}{2}+\frac{q_{1,2}^4(t)}{4}, \tag{5}$$

and plot these quantities in Fig. 3(a) for $p_1(0)=0.1$, $p_2(0)=0.2$ (the parameters for Fig. 1) by varying coupling ε. It is clear that there is a sharp transition at $\varepsilon=\varepsilon_c=0.0032$. Before ε_c $(\varepsilon<\varepsilon_c)$ there is a finite difference between h_1 and h_2, while above ε_c both oscillators have an identical average bare energy, clearly indicating MS. The finite jump of the average bare energy difference $\Delta h=h_2-h_1$ seems to indicate a first-order phase transition. In Fig. 3(b) we plot the largest Lyapunov exponent (LE) λ for the case of Fig. 3(a), and find zero λ before and after ε_c, justifying the MS transition is between quasiperiodic motions.

In Figs. 3(c) and 3(d) we do the same as for Figs. 3(a) and 3(b), respectively, but with $p_1(0)=5.42\times10^{-2}$ and $p_2(0)$

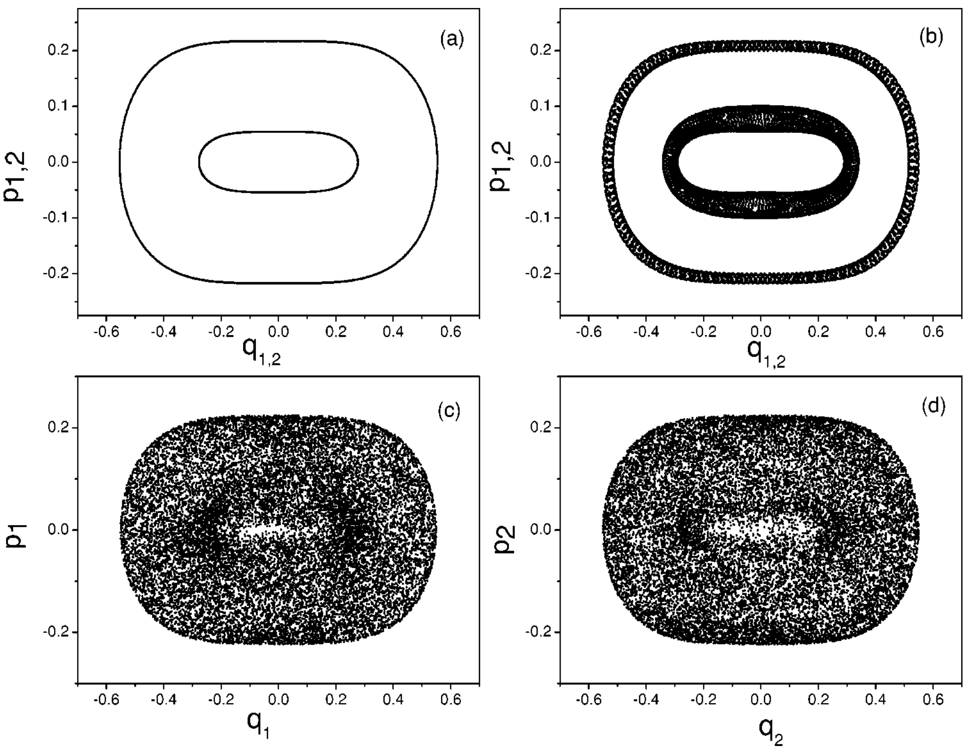

FIG. 2. The same as Fig. 1 with $p_1(0) = 5.42\times10^{-2}$ and $p_2(0) = 0.217$. (a) $\varepsilon = 0$, (b) $\varepsilon = 5.5\times10^{-3}$, (c) and (d) $\varepsilon = 1.5\times10^{-2}$. Measure synchronization is achieved in (c) and (d). The motion is periodic in (a), quasiperiodic in (b), and chaotic in (c) and (d).

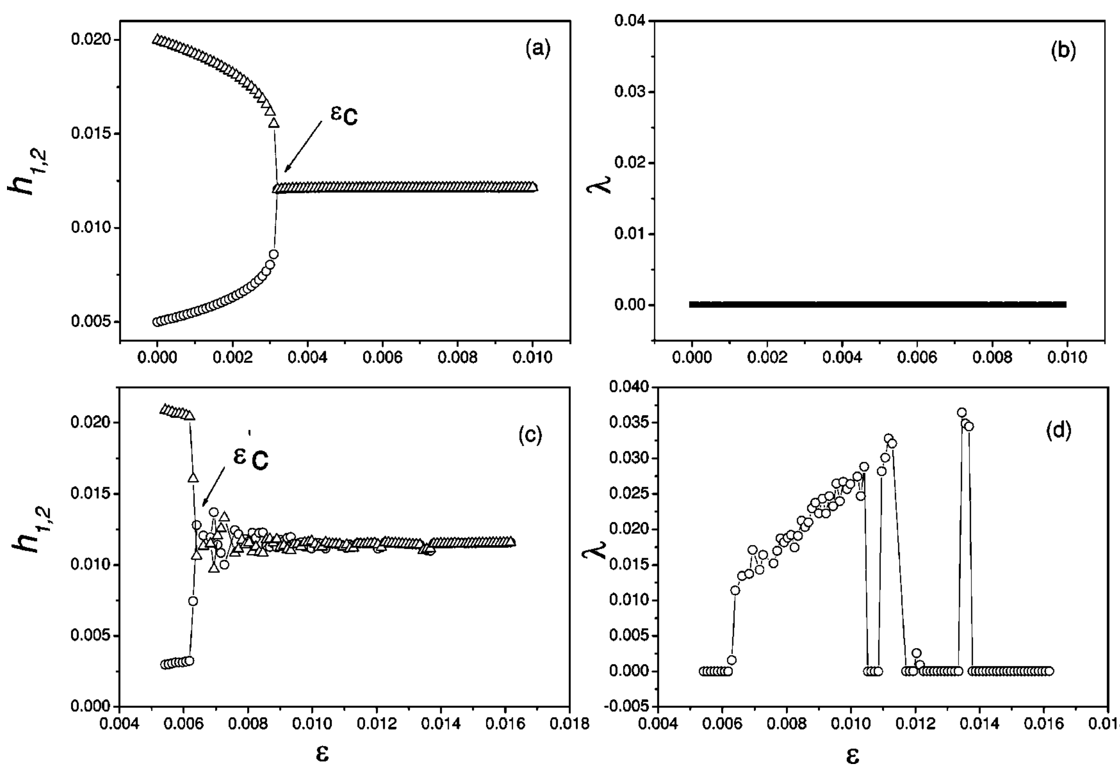

FIG. 3. (a) Average bare energies of Eqs. (5), $h_{1,2}$, plotted vs ε. $p_1(0) = 0.1$ and $p_2(0) = 0.2$. Measure synchronization occurs at a critical coupling $\varepsilon = \varepsilon_c = 0.0032$, where discontinuous jumps of $h_{1,2}$ can be identified. (b) The largest LE plotted vs ε. Zero λ before and after ε_c shows a MS transition from quasiperiodicity to quasiperiodicity. (c) Same as (a) with $p_1(0) = 5.42\times10^{-2}$ and $p_2(0) = 0.217$. MS occurs at $\varepsilon_c' = 6.3\times10^{-3}$ and the fluctuations of $h_{1,2}$ curves for $\varepsilon \gtrsim \varepsilon_c'$ are due to the chaos-induced long transient. (d) Same as (c) with the largest LE λ plotted. λ changes from zero to positive right at the MS critical point ε_c'. Thus, in this case the MS transition is associated with the transition from quasiperiodicity to chaos.

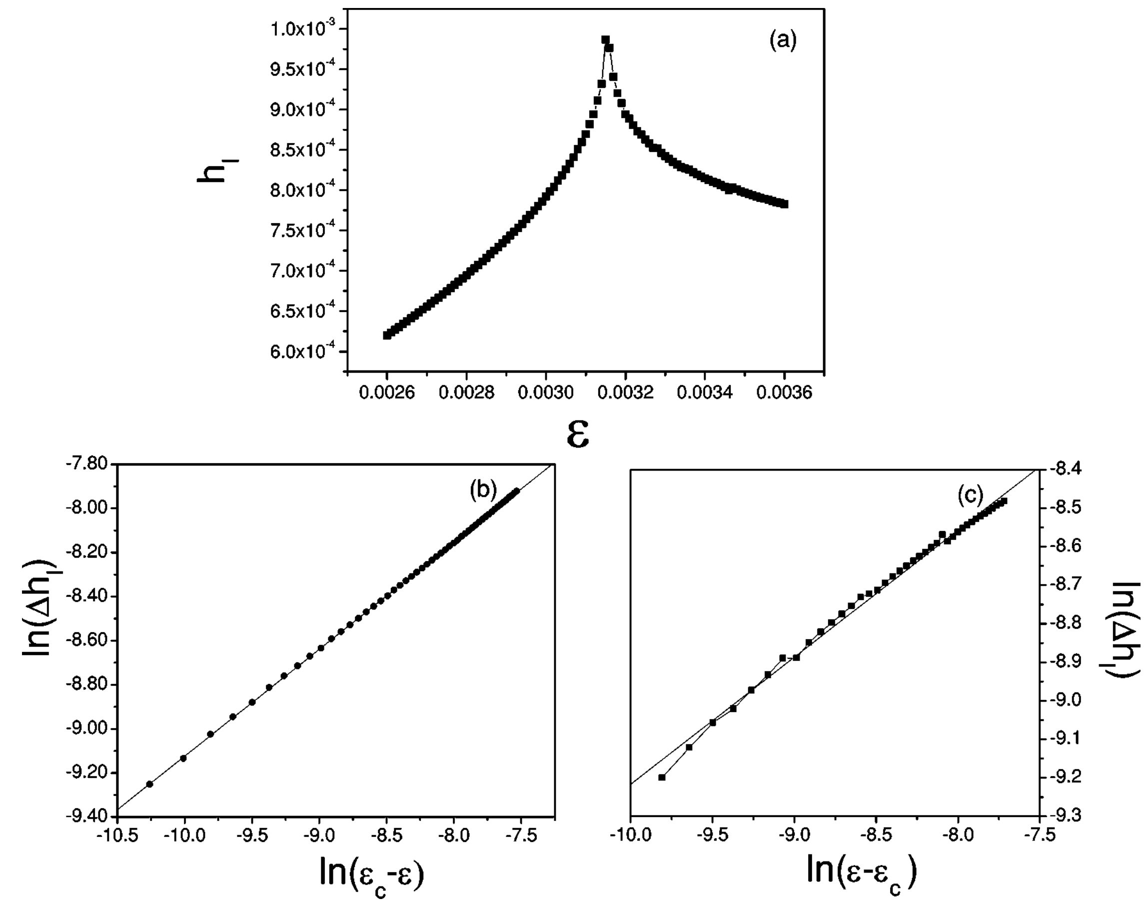

FIG. 4. $p_1(0)=0.1$ and $p_2(0)=0.2$ (for both Figs. 4 and 5). (a) Average interaction energy h_I of Eq. (6) plotted vs ε. Unlike Fig. 3(a), h_I is continuous and has a finite maximum value at ε_c. The derivative of the h_I curve has discontinuity at ε_c and has infinite slopes right before and after ε_c. (b) and (c) The power law scalings of $\Delta h_I(\varepsilon)=h_I(\varepsilon_c)-h_I(\varepsilon)$, Eq. (7), before and after ε_c, respectively.

$=0.217$ (the parameters of Fig. 2). From Fig. 3(c) we can see also a MS transition at $\varepsilon=\varepsilon_c'=6.3\times10^{-3}$, and from Fig. 3(d) we find zero λ (quasiperiodicity) for $\varepsilon<\varepsilon_c'$ and positive λ (chaoticity) for $\varepsilon>\varepsilon_c'$. Thus, the MS transition is directly associated to quasiperiodicity-chaoticity transition. This observation is quite different from the conclusion of Ref. [8], where the authors declared that MS transitions have no relation with transitions to chaos. For chaotic motion one needs extremely long time to reach $h_2=h_1$ for the MS state just after the MS transition, and this is the reason why $h_{1,2}$ fluctuate in Fig. 3(c) after ε_c'.

III. MS TRANSITION BETWEEN QUASIPERIODIC STATES

We now proceed to study the characteristic features of MS transitions. In this section, we focus on the QP to QP transition of Fig. 3(a). It is obvious that the interaction between the two φ^4 oscillators plays a key role for the MS transitions. Thus, it is interesting to investigate the behavior of the interaction energy in the transition process. In Fig. 4(a) we plot the average interaction energy

$$h_I=\frac{1}{T}\int_0^T \varepsilon[q_1(t)-q_2(t)]^2dt \tag{6}$$

against the coupling ε. We find that h_I has a singularity at the critical coupling ε_c. In Figs. 4(b) and 4(c) we show that this singularity has power law scalings

$$\begin{aligned} &h_I(\varepsilon_c)-h_I(\varepsilon)\propto(\varepsilon_c-\varepsilon)^{\alpha}, \quad \varepsilon<\varepsilon_c;\\ &h_I(\varepsilon_c)-h_I(\varepsilon)\propto(\varepsilon-\varepsilon_c)^{\beta}, \quad \varepsilon>\varepsilon_c;\\ &\alpha\approx\tfrac{1}{2}, \quad \beta\approx\tfrac{1}{3}. \end{aligned} \tag{7}$$

Several interesting features can be observed in Fig. 4. First, while the bare energies $h_{1,2}$ have discontinuity at ε_c in Fig. 3(a), h_I is continuous at the critical point in Fig. 4(a). Since $h_{1,2}$ and h_I have the same unit $(h_1+h_2+h_I=H=0.025)$, it is thus nontrivial that at the same critical point $h_{1,2}$ show a "first-order" phase transition while h_I shows a "second-order" phase transition. The MS transition seems to belong to a novel class of bifurcations.

Second, though h_I is continuous around ε_c and has a finite maximum value at ε_c, the derivative of h_I over ε [i.e., the slope of the $h_I(\varepsilon)$ curve] has a discontinuity and diverges at ε_c, following the scaling laws of Eq. (7). The shape of $h_I(\varepsilon)$ curve looks similar to the λ type of phase transition in statistical physics [14–16].

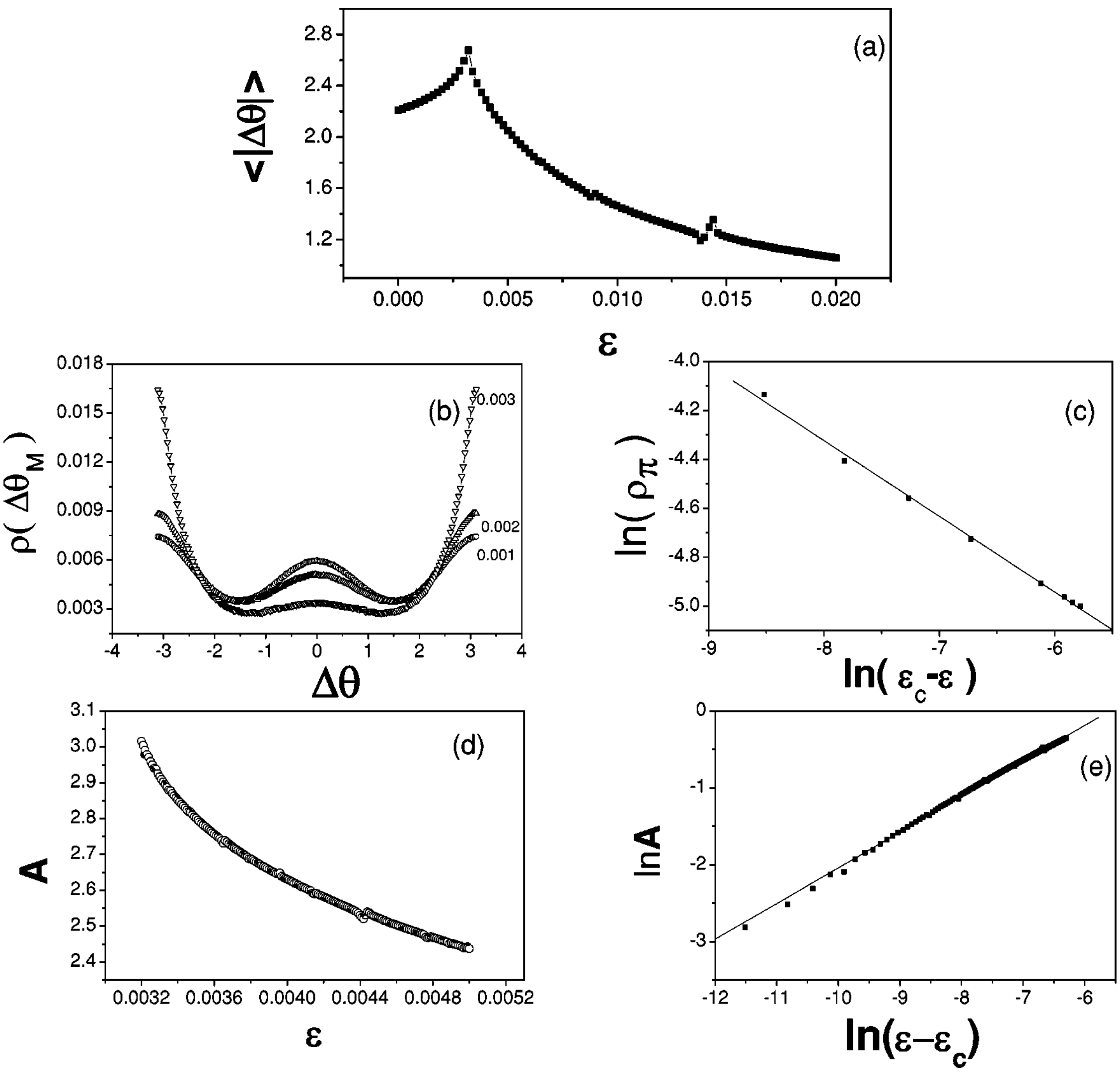

FIG. 5. (a) Average absolute phase difference $\langle|\Delta\theta|\rangle$ plotted vs ε. (b) Probability distribution density of $\Delta\theta_M$ for different couplings $\varepsilon<\varepsilon_c$. Singularity of $\rho(\Delta\theta_M)$ at $\Delta\theta_M=\pm\pi$ is observed. (c) The power law scaling of $\rho(\Delta\theta_M=\pm\pi)$ near the critical point $\varepsilon\lesssim\varepsilon_c$, Eq. (9). (d) The maximum phase difference (amplitude) $A=|\Delta\theta(t)|_{Max}$ plotted vs ε for $\varepsilon>\varepsilon_c$. Amplitude A decreases as ε increases, justifying the decreasing behaviors of h_I in Fig. 4(a) and $\langle|\Delta\theta|\rangle$ in Fig. 5(a) for $\varepsilon>\varepsilon_c$. (e) The power law scaling of Eq. (10) for the amplitude of phase difference for $\varepsilon>\varepsilon_c$.

It is interesting to understand why h_I increases so quickly before ε_c, while decreases slowly after the MS transition. Since the interaction energy is proportional to the coupling ε, it is thus natural that h_I increases as ε increases. However, a nontrivial behavior for $\varepsilon<\varepsilon_c$ is that h_I increases with ε much faster than linear increase. This feature can be explained, based on the average phase difference between the two oscillators defined as

$$\langle|\Delta\theta|\rangle=\frac{1}{T}\int_0^T|\Delta\theta(t)_M|dt,$$

$$\Delta\theta(t)_M=\mathrm{sgn}[\Delta\theta(t)][\pi-|\Delta\theta(t)\mathrm{mod}\ \pi|],$$

$$\Delta\theta(t)=\theta_1(t)-\theta_2(t),$$

$$\theta_i(t)=\arctan(p_i/q_i)\in[0,2\pi],\quad i=1,2. \tag{8}$$

Here, $\theta_{1,2}(t)$ is defined in the range $[0,2\pi]$ and $\Delta\theta(t)_M$ is defined in the range $[-\pi,\pi]$ so as to indicate the relative positions of two oscillators in the phase plane. We find in Fig. 5(a) that $\langle|\Delta\theta|\rangle$ increases as ε increases for $\varepsilon<\varepsilon_c$. This tendency causes the increase of the average spatial distance between the two oscillators, leading to a faster than linear increase of the interaction energy h_I with ε as showin in Fig. 4(a).

The behavior of the phase difference for $\varepsilon<\varepsilon_c$ can be understood as follows. The interaction is the key force for the measure synchronization. Before synchronization the two oscillators have different phase frequencies. Near the MS transition point, the difference of the frequencies of the two oscillators becomes small. At the same time, the phase difference has a larger probability of staying near the difference angles $\Delta\theta(t)_M\approx\pm\pi$ as ε is nearer to the critical coupling ε_c, because at these angles the interaction can play the most

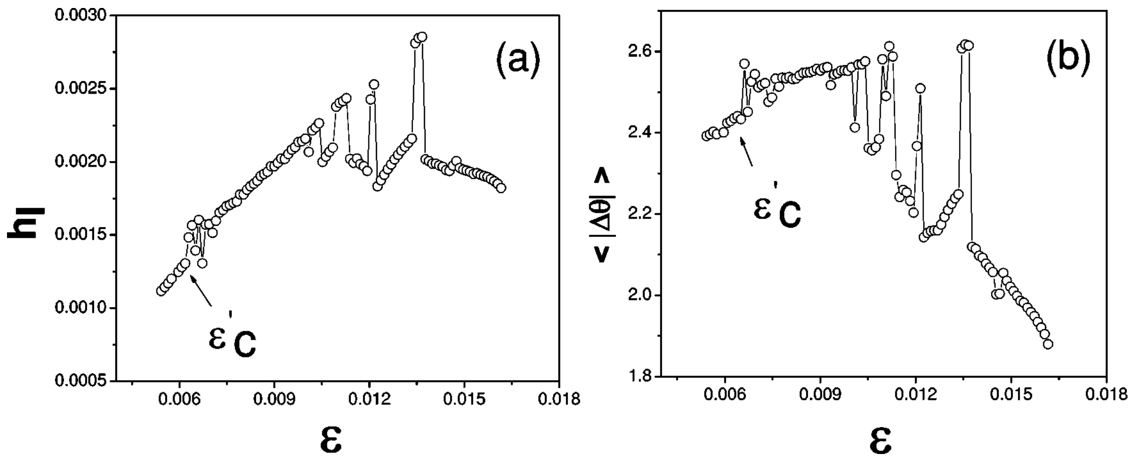

FIG. 6. $p_1(0)=5.42\times10^{-2}$ and $p_2(0)=0.217$ (for both Figs. 6 and 7). (a) Same as Fig. 4(a) but with a different initial condition. In the chaotic MS state, h_I monotonously increases with ε for $\varepsilon>\varepsilon_c'$ (excluding quasiperiodic states in some windows and for $\varepsilon\gg\varepsilon_c'$), in sharp contrast with the behavior of Fig. 4(a). (b) The average absolute phase difference $\langle|\Delta\theta|\rangle$ plotted against ε.

important role in bringing the two oscillators to synchronization and phase locking. This is why $\langle|\Delta\theta|\rangle$ increases with ε for $\varepsilon<\varepsilon_c$. In Fig. 5(b), we plot the phase difference probability distribution density $\rho(\Delta\theta_M)$ for different ε. It is clear that $\rho(\Delta\theta_M\approx\pm\pi)$ increases rapidly as ε approaches ε_c, and that $\rho(\Delta\theta_M\approx0)$ decreases as ε goes to ε_c. These explain well the tendencies of $\langle|\Delta\theta|\rangle$ in Fig. 5(a) and h_I in Fig. 4(a) for $\varepsilon<\varepsilon_c$. In Fig. 5(c), we show that $\rho(\Delta\theta_M\approx\pi)$ has a power law scaling relation with $\varepsilon_c-\varepsilon$ as

$$\rho(\Delta\theta_M\approx\pi)\propto(\varepsilon_c-\varepsilon)^{-1/3}. \quad (9)$$

After MS ($\varepsilon>\varepsilon_c$), the interaction energy h_I decreases with ε [Fig. 4(a)]. This phenomenon is a bit surprising since $E_I(t)$ appears to be proportional to ε. This tendency to decrease can be again understood, based on the behavior of the phase difference $\Delta\theta$. After the MS transition the phases of the two oscillators are already locked to each other with the difference $\Delta\theta(t)=\theta_1(t)-\theta_2(t)$ oscillating within the region of $(-\pi,\pi)$. In Fig. 5(d), we plot the variation of maximum phase difference $A=|\Delta\theta|_{Max}$ with coupling ε. The amplitude A decreases monotonicly from ε_c. The shrinking of $|\Delta\theta|$ leads to the decrease of h_I, and this decreasing trend is even more drastic than the linear increase of ε, leading to the overall decrease of h_I in Fig. 4(a) for $\varepsilon>\varepsilon_c$. Moreover, as ε approaches ε_c the slope of the A–ε curve diverges, while A itself approaches a finite value of π. This again shows a scaling power law relation of Fig. 5(e):

$$\pi-A(\varepsilon)\propto(\varepsilon-\varepsilon_c)^{1/2}. \quad (10)$$

IV. MS TRANSITION FROM QUASIPERIODICITY TO CHAOS

In the preceding section, we observed a MS transition from QP to QP and found the related power scaling laws. By changing the initial condition $(p_1(0),p_2(0))$ we can find the chaotic motion of the system. Chaos first appears around the critical coupling ε_c, and the chaotic region can be enlarged by further adjusting the initial state. For our system, we observe the transition from quasiperiodicity to chaos in association with the MS transition for certain initial conditions when ε is increased from zero, as shown in Figs. 3(c) and 3(d).

We now investigate the behaviors of the interaction energy h_I and the phase difference $\langle|\Delta\theta|\rangle$ also for the QP to the CH MS transition. Figure 6(a) is same as Fig. 4(a), but with the initial condition of Fig. 2, and find that the behavior of h_I is considerably different from that in Fig. 4(a) for the QP to the QP MS transition. First, there is no long any power law scaling [such as in Figs. 4(b) and 4(c)], and we observe a finite slope of the h_I curve for $\varepsilon\lesssim\varepsilon_c'$. This is not surprising since the scaling ε region around ε_c in Fig. 4 is now replaced by the chaos region for the present initial condition. The most important observation in Fig. 6(a) is that at the transition point h_I monotonously increases not only for $\varepsilon<\varepsilon_c'$, but also for $\varepsilon>\varepsilon_c'$ (excluding some fluctuation near ε_c' and some quasiperiodic windows in the chaotic region), and thus h_I does not take on a maximum at ε_c'. This is in sharp contrast with Fig. 4(a). In order to explain the monotonous increasing behavior of h_I, we again compute the phase difference $\langle|\Delta\theta|\rangle$ in Fig. 6(b), and find that $\langle|\Delta\theta|\rangle$ also increases monotonously with ε in the QP to CH MS transition. This observation coincides with the feature of Fig. 6(a), while dramatically differs from Fig. 5(a).

The dynamical mechanism underlying the monotonous behavior in Fig. 6 for h_I and $\langle|\Delta\theta|\rangle$ is that, unlike the quasiperiodic MS motion, the phases of the two oscillators can never be locked to each other in the chaotic MS state. In Fig. 7(a), we plot the time variation of $\tilde{\Delta}\theta(t)=\tilde{\theta}_1(t)-\tilde{\theta}_2(t)$ [different to Eq. (8), here $\tilde{\theta}_{1,2}(t)$ is defined in the real range] for a chaotic MS state. The figure shows clearly stick-slip motion [17–20], i.e., $\tilde{\Delta}\theta(t)$ oscillates within a 2π region in the long-time "synchronous" segment, and abruptly jumps to $2n\pi$ ($n=\pm1,\pm2,\dots$) angles in a short-time "desynchronous" segment, and then oscillates within the 2π region before repeating the desynchronous burst, and so on. The intervals of the synchronous segments and the jump distances of

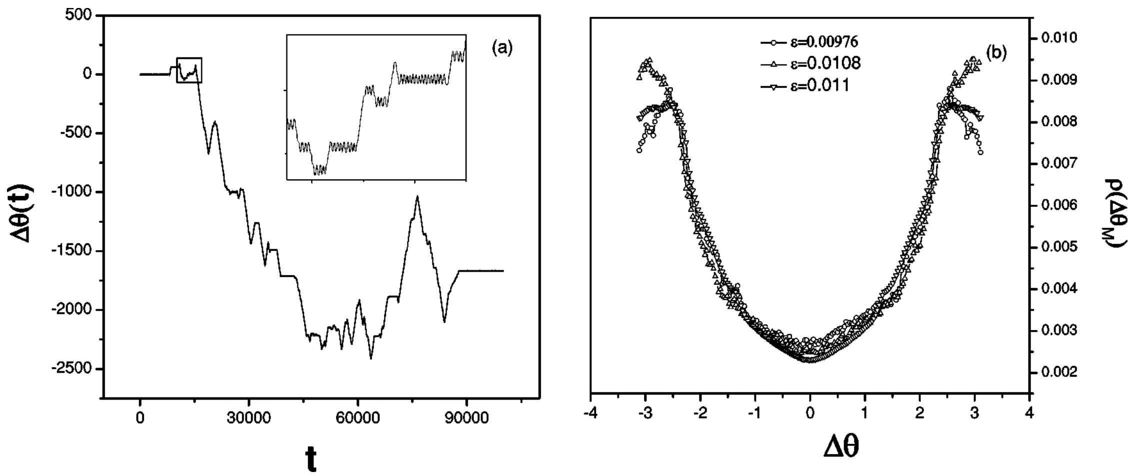

FIG. 7. (a) The stick-slip and random-walk-like motion of phase difference $\Delta\theta(t)$ in the chaotic MS state. (b) The probability distributions of $\Delta\theta_M$ for different couplings in the chaotic MS region.

the desynchronous segments appear to be random. An interesting fact is that the system can never reach full synchronization (i.e., a state with oscillation in 2π region only without any jump) in the chaotic MS states, no matter how large ε is. As ε increases in the chaotic MS region, more synchronous segments appear and longer phases are locked within 2π for longer periods of time. Also, with the stick-slip motion the phase difference $\Delta\theta(t)$ has a greater probability to stay around $\Delta\theta(t)_M \approx \pm\pi$ for larger coupling as shown in Fig. 7(b). This helps to explain why h_I increases monotonously in the chaotic region after MS.

V. DISCUSSION AND CONCLUSION

In conclusion we have investigated measure synchronization of two coupled φ^4 Hamiltonian oscillators for both quasiperiodic and chaotic motions. Certain quantities, such as the bare energies, the interaction energy, and the phase difference, are computed to quantitatively analyze the variations of the system dynamics and statistics. For the MS transition between quasiperiodic states, we find that while the average bare energy undergoes a discontinuous jump at the critical point, the average interaction energy is continuous and it takes the maximum value at the transition. However, the derivative of the interaction energy is discontinuous, and it diverges with certain power law scalings when the coupling approaches the critical point from both sides. The singular behavior of the interaction energy can be heuristically explained by the statistics of the phase difference of the two oscillators. For the chaotic MS state, we find that the phase difference performs a stick-slip and random-walk-like motion (long-time synchronous segments together with random desynchronous bursts). The average phase difference $\langle|\Delta\theta|\rangle$ increases with an increase of the coupling, and this leads to the monotonous increase of the interaction energy in the chaotic MS state.

It is emphasized that in both the quasiperiodic and chaotic measure-synchronous states, there are numerous other types of transitions to various quasiperiodic windows with different characteristic dynamics [see Figs. 5(a) and 3(d)], and it is interesting to clarify these transitions. In this paper, we have found some power law scalings at the MS transition from QP to QP numerically. Theoretical analysis has yet to be made for explaining these scaling laws for predicting the exponents and clarifying the relations between them. We hope to address these issues in a separate investigation.

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